Journal of Statistical Physics, Vol. 102, Nos. 3/4, 2001

Further Understanding of Hydrogen Atom: Yangian Approach and Physical Effect

Cheng-Ming Bai,[1] Mo-Lin Ge,[1, 2] and Kang Xue[1, 3]

Received April 10, 2000

By applying the representation theory of $Y(sl(2))$ to Hydrogen atom (HA) the correct spectrum are re-derived. This indicates the consistence between HA and the Yangian algebraic structure and guarantees that there is democracy between angular momentum **L** and Yangian current **J** in the sense of conserved currents. The physical effect of Yangian in HA has been predicted that preserves all the known results for HA, but gives rise to abnormal intensities in the spectrum lines near the free state.

KEY WORDS: Yangian; hydrogen atom; abnormal Zeeman effect.

1. INTRODUCTION

There is a close relationship between quantum mechanical models and Lie algebras for the existence of symmetries. It is well-known that Lie algebra describes the linear quantum space, i.e., quantum vector space. However, in many-body problems it is inevitable to meet tensor quantum space, for example, for interaction-spins located on different sites in a lattice. Hence, it is natural to extend Lie algebra to more general one. Among many possible candidates the preference is Yangian algebra presented by Drinfel'd[(1–3)] for the notable reasons:[(4)] (a) Yangian is related to RTT relation[(5–8)] that describes a large number of integrable models; (b) there appears natural connection between spectral parameter (viewed as one-dimensional momentum) and internal symmetry in Yangian representation; (c) the representation theory of Yangian had been well-established by Chari–Pressley.[(9)]

[1] Theoretical Physics Division, Nankai Institute of Mathematics, 300071 Tianjin, People's Republic of China.
[2] Center for Advanced Study, Tsinghua University, 100084 Beijing, People's Republic of China.
[3] Phys. Dept. of Northeast University, 130024 Chang-Chun, People's Republic of China.

0022-4715/01/0200-0545$19.50/0

As far as the physical applications of Yangian are concerned, besides the models with Yangian symmetry,[(10–14)] a simple example is Hydrogen atom (H-A). Since Pauli, people are familiar with the $so(4)$ symmetry for H-A, i.e., there are two conserved vectors—angular momentum $\mathbf{L}$ and Pauli–Runge–Lenz vector $\mathbf{A}=\frac{1}{2}(\mathbf{L}\times\mathbf{P}-\mathbf{P}\times\mathbf{L})+\kappa\mathbf{r}/r$ where $\mathbf{P}$ the momentum and κ the charge. Both of them commute with the Hamiltonian $H_0=\frac{1}{2}\mathbf{P}^2-\kappa/r$. For the bound state, putting $\mathbf{B}=1/\sqrt{-2H_0}\,\mathbf{A}$, then $\mathbf{L}$ and $\mathbf{B}$ form the relations:

$$\begin{aligned}
&[L_\lambda, L_\mu]=i\varepsilon_{\lambda\mu\nu}L_\nu \\
&[L_\lambda, B_\mu]=i\varepsilon_{\lambda\mu\nu}B_\nu \qquad (\lambda, \mu, \nu=1,2,3) \\
&[B_\lambda, B_\mu]=i\varepsilon_{\lambda\mu\nu}L_\nu
\end{aligned} \tag{1.1}$$

that lead to $so(4)$ symmetry by introducing

$$\mathbf{I}_1=\tfrac{1}{2}(\mathbf{L}+\mathbf{B}), \qquad \mathbf{I}_2=\tfrac{1}{2}(\mathbf{L}-\mathbf{B}) \tag{1.2}$$

satisfying

$$[I_{i\lambda}, I_{j\mu}]=i\varepsilon_{\lambda\mu\nu}\,\delta_{ij}I_{i\nu} \qquad (i=1,2) \tag{1.3}$$

For H-A without monopole

$$\mathbf{L}\cdot\mathbf{B}=\mathbf{B}\cdot\mathbf{L}=0 \tag{1.4}$$

hence, one has

$$\mathbf{I}_1^2=\mathbf{I}_2^2=-\tfrac{1}{4}\left(\tfrac{1}{2}H_0^{-1}\kappa^2+1\right) \tag{1.5}$$

If we denote by $k(k+1)$ the eigenvalues of $\mathbf{I}_1^2=\mathbf{I}_2^2$, $(k=0, \frac{1}{2}, 1,\ldots)$ from Eq. (1.5) it follows immediately the spectrum of H-A:

$$E_n=-\frac{\kappa^2}{2n^2} \qquad (n=2k+1) \tag{1.6}$$

All the above historical derivations are based on the properties of vector space. However, actually the H-A is described in terms of "spinors" $\mathbf{I}_1$ and $\mathbf{I}_2$. The vector $\mathbf{L}$ obeys

$$\mathbf{L}=\mathbf{I}_1+\mathbf{I}_2 \tag{1.7}$$

and eigenvalues of $\mathbf{L}^2 = l(l+1)$ are $l = 2k, 2k-1,\dots, 0$. In general, l can be written as

$$l = 2k - p, \qquad p = 0, 1,\dots, 2k \tag{1.8}$$

Therefore, essentially H-A takes quantum tensor space formed by $\mathbf{I}_1$ and $\mathbf{I}_2$. Since H-A possesses Yangian symmetry,[15] it is natural to ask whether the representation theory of $Y(sl(2))$ guarantees the correct spectrum given by Eq. (1.6). This is highly nontrivial, because Yangian is much larger than Lie algebra and the representations of Yangian are completely different from those of Lie algebras. In this paper we would like to show the following new results:

(a) the representation of Yangian working in tensor space gives the correct spectrum of H-A;

(b) the conserved Yangian current is naturally introduced that it is more general than the usual angular momentum **L**.

(c) the physical effect of Yangian in H-A is presented that preserves all the known theoretical and experimental results of H-A, but changes the intensity of the spectra near the free states at large n and l where n and l are principle and angular momentum quantum numbers, respectively.

For the self-contained we first introduce the main conclusions of representations of $Y(sl(2))$, then apply them to H-A without and with monopole. Finally, the prediction of a new sort of abnormal Zeeman effect due to Yangian is presented.

2. *Y(sl(2))* AND EIGENVALUES OF $\mathbf{J}^2$

Yangian algebras were established by Drinfel'd in his three original papers which are referred to refs. 1–3 for details. A Yangian is formed by a set $\{\mathbf{I}, \mathbf{J}\}$ obeying the commutation relations (together with the co-product given in ref. 1),

$$[I_\lambda, I_\mu] = c_{\lambda\mu\nu} I_\nu, \qquad [I_\lambda, J_\mu] = c_{\lambda\mu\nu} J_\nu \tag{2.1}$$

$$[J_\lambda, [J_\mu, I_\nu]] - [I_\lambda, [J_\mu, J_\nu]] = h^2 a_{\lambda\mu\nu\alpha\beta\gamma} \{I_\alpha, I_\beta, I_\gamma\} \tag{2.2}$$

$$[[J_\lambda, J_\mu], [I_\sigma, J_\tau]] + [[J_\sigma, J_\tau], [I_\lambda, J_\mu]]$$
$$= h^2 (a_{\lambda\mu\nu\alpha\beta\gamma} c_{\sigma\tau\nu} + a_{\sigma\tau\nu\alpha\beta\gamma} c_{\lambda\mu\nu}) \{I_\alpha, I_\beta, J_\gamma\} \tag{2.3}$$

where the set of I_λ forms a simple Lie algebra characterized by $c_{\lambda\mu\nu}$ and the repeated indices mean summation. The definitions of $a_{\lambda\mu\nu\alpha\beta\gamma}$ and $\{x_i, x_j, x_k\}$ were given in ref. 1,

$$
\begin{aligned}
a_{\lambda\mu\nu\alpha\beta\gamma} &= \frac{1}{4!} c_{\lambda\alpha\sigma} c_{\mu\beta\tau} c_{\nu\gamma\rho} c_{\sigma\tau\rho} \\
\{x_1, x_2, x_3\} &= \sum_{i \neq j \neq k} x_i x_j x_k \qquad \text{(symmetric summation)}
\end{aligned} \tag{2.4}
$$

and through the mapping shown in ref. 3, Eqs. (2.1)–(2.3) can be connected with the "new realization of a Yangian" defined in ref. 3. For $Y(sl(2))$ the Eqs. (2.1)–(2.3) can be simplified.

Firstly, both sides of Eq. (2.2) are zeros and (h an arbitrary constant) with the notations $I_\pm = I_1 \pm iI_2$, $J_\pm = J_1 \pm iJ_2$, a direct calculation verifies that Eq. (2.1) can be recast as

$$
[I_3, I_\pm] = \pm I_\pm, \qquad [I_+, I_-] = 2I_3 \tag{2.5}
$$

$$
\begin{aligned}
[I_3, J_\pm] &= [J_3, I_\pm] = \pm J_\pm \\
[I_+, J_-] &= [J_+, I_-] = 2J_3
\end{aligned} \tag{2.6}
$$

Furthermore Eq. (2.3) becomes

$$
[I_\pm, [J_3, J_\pm]] = \tfrac{1}{4} h^2 I_\pm (I_3 J_\pm - J_3 I_\pm) \tag{2.7}
$$

that together with Eqs. (2.5–2.6) and Jacobian identities yield all of the relations in Eq. (2.3), i.e., most of ones in Eq. (2.3) are not independent. The algebraic meaning of $Y(sl(2))$ is clear that it contains $sl(2)$ shown by Eq. (2.5) as a subalgebra. The other three new generators $\mathbf{J}$ are beyond $sl(2)$ itself and obey Eq. (2.6) and nonlinear commutation relations Eq. (2.7). There are many physical realizations of $Y(sl(2)) = Y(\mathbf{I}, \mathbf{J})$ satisfying Eq. (2.5)–Eq. (2.7) that can be made in terms of spin-chain and in elementary Quantum Mechanics, see refs. 10–14, 16.

There has been mathematical theory of representation of $Y(sl(2))$ referring to Chari and Pressley.[9] To follow the theory let us consider the tensor basis in $sl(2)$-algebraic space

$$
\Omega_p = \sum_{i=0}^{p} (-1)^i \frac{(m-i)!\,(n-p+i)!}{m!\,(n-p)!} (e_{m-i} \otimes e_{n-p+i}) \tag{2.8}
$$

where the coefficients essentially are nothing but the Clebsch–Gordan coefficients and $e_m \equiv e_{m,m}$ is basis of $sl(2)$ with the highest third component for a given weight $m/2$. For convenience the notations in ref. 9 have been used. The relationship between e_m and those which have been familiar in Physics is the following: $m = 2j$ and for $M \geqslant 1$

$$e_{j,\,j-M} = \left[\frac{j(j-1/2)\cdots(j-M/2+1/2)}{M!} \right]^{1/2} \psi_{j,\,j-M} \tag{2.9}$$

$$e_m \equiv e_{m,\,m} \leftrightarrow e_{j,\,j} = \psi_{j,\,j} \tag{2.10}$$

The Ω_p is eigenstate of operator $\mathbf{J}^2 = \mathbf{J} \cdot \mathbf{J}$. There is big difference between eigenvalues of $\mathbf{J}^2$ and $\mathbf{I}^2$ for the following reason. Suppose $\mathbf{I}$ and $\mathbf{J}$ satisfy Yangian relations Eq. (2.5)–Eq. (2.7), so do $\mathbf{I}$ and $\lambda\mathbf{I} + \mathbf{J}$ where λ is an arbitrary parameter. It is called "translation" of Yangian. Furthermore, it turns out that the set

$$\mathbf{I} = \sum_{i=1}^{N} \mathbf{I}_i \tag{2.11}$$

$$\mathbf{J} = \left(\frac{ih}{8}\right) \sum_{i \neq j}^{N} \varepsilon_{ij} \mathbf{I}_i \times \mathbf{I}_j \qquad \left(\varepsilon_{ij} = \begin{matrix} 1 & (i > j) \\ 0 & (i = j) \\ -1 & (i < j) \end{matrix} \right) \tag{2.12}$$

satisfies $Y(sl(2))$, i.e., Eq. (2.5)–Eq. (2.7) where i can either specify particle or lattice site. The direct check verifies that the "local translation" of $\mathbf{J}$ given by Eq. (2.12) still satisfies $Y(sl(2))$:

$$\begin{aligned} \mathbf{I} &= \sum_{i=1}^{N} \mathbf{I}_i \\ \mathbf{J} &= \frac{Qh}{4} \left(\sum_{i=1}^{N} \mu_i \mathbf{I}_i + \frac{i}{2} \sum_{i \neq j}^{N} \varepsilon_{ij} \mathbf{I}_i \times \mathbf{I}_j \right) \end{aligned} \tag{2.13}$$

where μ_i's are arbitrary (complex) constants. Equation (2.13) indicates that $\mathbf{J}$ is spectral parameters dependent (μ_i) operator, so that the representation of $Y(\mathbf{I}, \mathbf{J})$ should depend on the spectral parameter that is completely different from the Lie algebraic one. The physical meaning of such a dependence can be understood in such way that in Yangian description the internal degree of freedom is connected with the spatial behavior, for example, μ_i may be momentum or phase shift along a lattice. As usual, Yangian is related to integrable models where the spin flip often gives rise to change one-dimensional momentum (spectral parameter) in Bethe ansatz problem.

For simplicity we consider the case $N=2$, then Eq. (2.13) is written in the form

$$\mathbf{I}=\mathbf{I}_1+\mathbf{I}_2 \tag{2.14}$$

$$\mathbf{J}=\frac{ih}{4}(a\mathbf{I}_1+b\mathbf{I}_2+\mathbf{I}_1\times\mathbf{I}_2) \tag{2.15}$$

where a and b are arbitrary constants. Acting $\mathbf{J}^2$ on Ω_p we find eigenvalue

$$\mathbf{J}^2\Omega_p=\mathscr{I}_{m,n}(a,b;p)\,\Omega_p \tag{2.16}$$

where

$$\begin{aligned}\mathscr{I}_{m,n}(a,b;p)={}&m\left(\frac{m}{2}+1\right)a^2+n\left(\frac{n}{2}+1\right)b^2\\&+[(m-2p)(n-2p)-2p(p+1)]\,ab\\&-\frac{1}{4}ml(m+n)-\frac{p}{2}(m+n-1-p)\\&\times[mn+2-p(m+n+1-p)]\end{aligned} \tag{2.17}$$

Noting that in $\mathscr{I}(a,b;p)$ there appears the "interference" term coming from the overlapping between space 1 occupied by $\mathbf{I}_1$ and space 2 by $\mathbf{I}_2$. This is because $\mathscr{I}(a,b;p)$ comes from the co-product of operator $\mathbf{J}$.[9] In general, $\mathbf{J}^2$ is not necessarily Hermitian.

Equipping with the above knowledge we come to employ the Yangian theory to H-A.

3. REPRESENTATION OF *Y(sl(2))* AND SPECTRUM OF HYDROGEN ATOM

As was pointed out in ref. 15 that for H-A the set formed by

$$\mathbf{I}=\mathbf{L}=\mathbf{I}_1+\mathbf{I}_2 \tag{3.1}$$

and

$$\mathbf{J}=\frac{ih}{4}\mathbf{L}\times\mathbf{B} \tag{3.2}$$

satisfies $Y(sl(2))$ and $[H_0, \mathbf{I}] = [H_0, \mathbf{J}] = 0$ i.e., $[H_0, Y(sl(2))] = 0$, where $\mathbf{B} = 1/\sqrt{-2H_0}\,\mathbf{A}$ and $\mathbf{A}$ is Pauli–Runge–Lenz vector. In comparison to Eq. (2.15) on account of Eq. (1.2) we have

$$a = -1, \qquad b = 1 \tag{3.3}$$

Recalling the eigenvalue formula of $\mathbf{J}^2$ Eq. (2.17) for any a and b it can be recast to

$$\begin{aligned}\mathscr{I}_{j_1,\, j_2}(p) = {} & \frac{h^2}{16}\{j_1(j_1+1)\,a^2 + j_2(j_2+1)\,b^2 \\ & + [2(j_1-p)(j_2-p) - p(p+1)]\,ab\} \\ & - \frac{h^2}{4}\left\{(j_1+j_2)\,j_1 j_2 + \frac{p}{4}(2j_1+2j_2+1-p)\right. \\ & \left. \times [4j_1j_2 + 2 - p(2j_1+2j_2+1-p)]\right\}\end{aligned} \tag{3.4}$$

where $m = 2j_1$, $n = 2j_2$ have been used.

For $j_1 = j_2 = k$ and $b = -a = 1$, Eq. (3.4) is simplified to

$$\mathscr{I}_k = \left(+\frac{h^2}{16}\right)(2k-p)(2k-p+1)[-4k(k+1) + (2k-p)(2k-p+1) + 1] \tag{3.5}$$

We emphasize that Eq. (3.5) is completely the consequence of representation of $Y(sl(2))$ for $b = -a = 1$ and $j_1 = j_2 = k$ ($\mathbf{I}_1^2 = j_1(j_1+1)$) and $\mathbf{J}^2$ only works in tensor space.

Now let us turn to H-A. On account of Eq. (3.2) it is easy to find

$$\mathbf{J}^2 = \left(-\frac{h^2}{16}\right)\mathbf{L}^2(\mathbf{B}^2 - 1) \tag{3.6}$$

Noting that

$$\mathbf{L}^2 = \mathbf{I}_1^2 + \mathbf{I}_2^2 + 2\mathbf{I}_1 \cdot \mathbf{I}_2 \tag{3.7}$$

whose eigenvalue is $l(l+1)$, whereas

$$\mathbf{B}^2 = \mathbf{I}_1^2 + \mathbf{I}_2^2 - 2\mathbf{I}_1 \cdot \mathbf{I}_2 \tag{3.8}$$

whose eigenvalue is $4k(k+1)-l(l+1)$ for H-A without monopole which leads to $\mathbf{I}_1^2=\mathbf{I}_2^2=k(k+1)$. Of course, since $\mathbf{L}=\mathbf{I}_1+\mathbf{I}_2$, the eigenvalues of l take $(2k+1)$ values: $l=2k-p$, $(p=0, 1,..., 2k)$. Substituting Eq. (3.7) and Eq. (3.8) into Eq.(3.6) it follows the eigenvalue of $\mathbf{J}^2$:

$$\mathscr{I}=\left(\frac{h^2}{16}\right) l(l+1)[-4k(k+1)+l(l+1)+1] \tag{3.9}$$

On the other hand the direct calculation of $\mathbf{J}^2$ in terms of $\mathbf{L}$ and $\mathbf{B}$ gives

$$\mathbf{J}^2=\frac{h^2}{16}\mathbf{L}^2\left(\mathbf{L}^2+\frac{1}{2}H_0^{-1}\kappa^2+2\right) \tag{3.10}$$

whose eigenvalue

$$\mathscr{I}=\frac{h^2}{16}l(l+1)\left[l(l+1)+\frac{\kappa^2}{2E}+2\right] \tag{3.11}$$

Identifying Eq. (3.9) with Eq. (3.11) it yields

$$E=-\frac{\kappa^2}{2(2k+1)^2}=-\frac{\kappa^2}{2n^2} \qquad \left(k=0, \frac{1}{2}, 1,...\right) \tag{3.12}$$

Substituting $l=2k-p$ into Eq. (3.11) the eigenvalues of $\mathbf{J}^2$ on the basis of direct calculation for H-A are given by

$$\mathscr{I}=\frac{h^2}{16}(2k-p)(2k-p+1)[(2k-p)(2k-p+1)+1-4k(k+1)] \tag{3.13}$$

that is exactly the same as Eq. (3.5) given by Yangian representation independently.

It may argue that since the spectrum are completely determined by $\mathbf{J}^2$ from the point of view of Yangian, whereas $\mathbf{J}$ subjects to an arbitrary translation, i.e., $\mathbf{J}\to\mathbf{J}+\lambda\mathbf{I}$ still preserves $Y(sl(2))$, the translation term $\lambda\mathbf{I}$ may change the spectrum Eq. (3.12). The answer is negative. The direct calculation shows that such a translation does not change the spectrum at all.

Therefore based on the tensor space of $\mathbf{I}_1$ and $\mathbf{I}_2$ i.e., $so(4)$ space we have shown that the "current" $\mathbf{J}$ provides the correct spectrum of H-A in terms of the representation of Yangian. The deep reason for this is H-A is related to RTT relation[16] which is also the origin of Yangian, otherwise it is almost not possible to meet such a nice consistence.

4. *Y(sl(2))* AND HYDROGEN-LIKE ATOM WITH MONOPOLE

When Hydrogen-like atom possesses monopole the Hamiltonian reads

$$H=\frac{\pi^2}{2\mu}+\frac{1}{2\mu}\frac{q^2}{r^2}-\frac{\kappa}{r}, \qquad \pi=p-zeA \tag{4.1}$$

where μ is mass, $q=zeg$, $\kappa=ze^2$ and g being monopole charge. The corresponding mechanical angular momentum and rescaled Pauli–Runge–Lenz vector are

$$\mathbf{L}'=\frac{1}{2}(\mathbf{r}\times\pi-\pi\times\mathbf{r})-q\frac{\mathbf{r}}{r}, \qquad \mathbf{B}'=\frac{i}{\sqrt{2\mu H}}\mathbf{R}' \tag{4.2}$$

$$\mathbf{R}'=\frac{1}{2}(\pi\times\mathbf{L}'-\mathbf{L}'\times\pi)-\frac{\mu\kappa}{r}\mathbf{r} \tag{4.3}$$

$$\mathbf{I}'_1=\frac{1}{2}(\mathbf{L}'+\mathbf{B}'), \qquad \mathbf{I}'_2=\frac{1}{2}(\mathbf{L}'-\mathbf{B}') \tag{4.4}$$

In difference from H-A without monopole it holds[16, 17]

$$\mathbf{L}'\cdot\mathbf{B}'=\mathbf{B}'\cdot\mathbf{L}'=\mathbf{I}'^2_1-\mathbf{I}'^2_2=q\sqrt{-\frac{\mu\kappa}{2H}} \tag{4.5}$$

when $q\neq 0$ we have

$$\mathbf{I}'^2_1=\frac{1}{4}\left[\left(q+\sqrt{-\frac{\mu\kappa}{2H}}\right)^2-1\right] \tag{4.6}$$

where

$$q=|j_1-j_2| \tag{4.7}$$

Since $\mathbf{I}'_1$ and $\mathbf{I}'_2$ still obey $so(4)$ as $\mathbf{I}_1$ and $\mathbf{I}_2$ do, we shall do not distinguish the quantum numbers for $\mathbf{I}'^2$ from those for $\mathbf{I}^2$. We find

$$E=-\frac{\mu\kappa^2}{2}\frac{1}{n^2}, \qquad n=j_1+j_2+1 \tag{4.8}$$

where $\mathbf{I}'^2_i=j_i(j_i+1)$ $(i=1,2)$. It turns out that

$$l=j_1+j_2-p=n-p-1 \tag{4.9}$$

where

$$\mathbf{L}'^2 = l(l+1), \quad l = \begin{cases} 0, 1, \ldots, n-1 & n = \text{integer} \\ 1/2, 3/2, \ldots, n-1 & n = \text{half integer} \end{cases} \tag{4.10}$$

and

$$q = \begin{cases} 0, 1, \ldots, n-1 & n = \text{integer} \\ 1/2, 3/2, \ldots, n-1 & n = \text{half integer} \end{cases} \tag{4.11}$$

The Eq. (4.11) does not mean that q should be equal to l, rather, for example,

$$n = 1, \quad j_1 = j_2 = 0: l = q = 0$$

$$n = \tfrac{3}{2}, \quad j_1 = \tfrac{1}{2}, j_2 = 0; j_1 = 0, j_2 = \tfrac{1}{2}: l = q = \tfrac{1}{2}$$

$$n = 2, \quad j_1 = 1, j_2 = 0 \quad \text{or} \quad j_1 = 0, j_2 = 1: l = 1, q = 1$$

$$j_1 = j_2 = \tfrac{1}{2}: l = 1, 0, q = 0$$

$$n = \tfrac{5}{2}, \quad j_1 = \tfrac{3}{2}, j_2 = 0, \quad \text{or} \quad j_1 = 0, j_2 = \tfrac{3}{2}; l = \tfrac{3}{2}, q = \tfrac{3}{2}$$

$$j_1 = 1, j_2 = \tfrac{1}{2} \quad \text{or} \quad j_1 = \tfrac{1}{2}, j_2 = 1: l = \tfrac{3}{2}, \tfrac{1}{2}, q = \tfrac{1}{2}$$

obviously, this is because l stands for the quantum number related to the vector sum of $\mathbf{I}'_1 + \mathbf{I}'_2$, whereas q is the difference of $\mathbf{I}'_1$ and $\mathbf{I}'_2$, but there are degeneracies.

In parallel to Section 2, for H-A with monopole we have

$$\mathbf{J}'^2 = \left(-\frac{h^2}{16}\right) \{\mathbf{L}'^2(\mathbf{B}'^2 - 1) - (\mathbf{L}' \cdot \mathbf{B}')^2\} \tag{4.12}$$

and

$$\mathbf{B}'^2 = -\left(\mathbf{L}'^2 + \frac{\mu\kappa}{2H} + 1\right) + q^2 \tag{4.13}$$

hence

$$\mathbf{J}'^2 = \frac{h^2}{16} \left\{\mathbf{L}'^4 + \frac{\mu\kappa^2}{2H} (\mathbf{L}'^2 - q^2) + (2 - q^2)\, \mathbf{L}'^2\right\} \tag{4.14}$$

Recalling $j_1 + j_2 = l + p$ $(p = 0, 1, \ldots, j_1 + j_2)$ we obtain eigenvalues of $\mathbf{J}'^2$:

$$\mathscr{I}' = -\frac{h^2}{16}\{[(l+1)(2p+1)+p^2][l(l+1)-q^2]-2l(l+1)\} \tag{4.15}$$

After computation Eq. (4.15) can be recast to

$$\begin{aligned}\mathscr{I}' = \frac{h^2}{16}\{&j_1(j_1+1)+j_2(j_2+1)-[2(j_1-p)(j_2-p)-p(p+1)]\\ &-4(j_1+j_2)\,j_1 j_2 - p(2j_1+2j_2+1-p)\\ &\times[4j_1 j_2+2-p(2j_1+2j_2+1-p)]\}\end{aligned} \tag{4.16}$$

which is exactly the same as Eq. (3.4) for $b = -a = 1$. When there is monopole the commuting set $\{H_0, \mathbf{L}^2, L_3\}$ is enlarged to the set $\{H, \mathbf{L}^2, L_3, \mathbf{J}^2\}$. For fixed n on substituting $p = n-1-l$ into Eq. (4.15) we obtain

$$\begin{aligned}\mathscr{I}' = \left(-\frac{h^2}{16}\right)\{&[(l+1)(2(n-1-l)+1)+(n-1-l)^2][l(l+1)-q^2]\\ &-2l(l+1)\}\end{aligned} \tag{4.17}$$

It is interesting to point out that eigenstate Ω_p is related to the monopole-spherical harmonic presented by Wu and Yang.[18]

In conclusion, if there is monopole it allows $j_1 \neq j_2$ $(|j_1 - j_2| = q)$, the eigenvalue of $\mathbf{J}'^2$ still completely determines the spectrum of H-A with monopole and coincides with the representation of $Y(sl(2))$ for $j_1 \neq j_2$ (i.e., $m \neq n$ in Eq. (2.8)).

5. YANGIAN EFFECT IN HYDROGEN ATOM

For H-A (without monopole) we have shown that the "conserved current" $\mathbf{L}$ can be extended to

$$\mathbf{J} = \lambda\mathbf{L} + \frac{ih}{4}\mathbf{L}\times\mathbf{B} \tag{5.1}$$

where λ is an arbitrary constant. For $h = 0$ it reduces to the usual angular momentum and for $h \neq 0$ $\mathbf{J}^2$ generates the same eigenvalues of energy. The extended current $\mathbf{J}$ is conserved

$$[H_0, \mathbf{J}] = 0 \tag{5.2}$$

and acts on the tensor space in $so(4)$. The term $\mathbf{L}\times\mathbf{B}$ represents a new type of interaction acting on tensor space.

Now, is there new effect of $\mathbf{J}$ other than the re-determination of correct spectrum for H-A only? Theoretically, the answer is yes, even the effect is so small that it does not change all the known results for H-A and hard to observe.

The predicted effect of $\mathbf{J}$ is to find Zeeman effect for $\mathbf{J}$, rather that $\mathbf{L}$ itself. The degeneracy for $\mathbf{J}$ will be removed by applied magnetic field $\mathscr{B}=B_0 e_z$ where B_0 is constant. The interaction Hamiltonian

$$H_I=\mu\mathscr{B}\cdot\mathbf{J}=\mu B_0 J_3 \tag{5.3}$$

where $\mathbf{J}$ is given by Eq. (5.1).

The transition caused by H_I consists in computing

$$\varepsilon=\langle\psi_{n',l',m'}|\,H_I\,|\psi_{n,l,m}\rangle \tag{5.4}$$

where $\psi_{n,l,m}$ is the eigenfunction of H-A. We observe that the interaction $\mathscr{B}\cdot(\mathbf{L}\times\mathbf{B})$ breaks the time-reversal invariance, so that it gives rise to very small contribution to $\langle H_I\rangle$. This is the reason why the perturbation can be used well. Nothing is surprise to appear such an effect under an applied magnetic field. There is often time-reversal broken effect in Chern–Simons type of interaction.

The energy correction besides the usual Zeeman effect ($\mathbf{E}_l\sim\mu B_0 L_3$) can be found:

$$\begin{aligned}\varepsilon_l(\text{Yangian})&=(\text{const})\,n\omega_l\{l'(l'+1)+l(l+1)\\&\quad+2\kappa(\bar{r}_{n,l'}+\bar{r}_{n,l})\}\langle n,l'|\,x_3\,|n,l\rangle\end{aligned} \tag{5.5}$$

where $l'=l+1$, $\omega_l=((\varepsilon_{l'}-\varepsilon_l)/h)$ and $\bar{r}_{n,l}$ stands for the average radius at the (n,l)-orbit. The bracket represents the dipole transition between l and $l'=l+1$ states. Since the effect is very small we should take n and l as larger as they could. The current experiments can reach $n\leqslant 100$, correspondingly we should take large l so that $l'\approx l$, hence

$$\varepsilon_l(\text{Yangian})\sim nl^2\cdot\text{dipole transition} \tag{5.6}$$

that occurs near the free state. It is worth to note that such a correction does not change all of the spectrum. It only changes the intensities of the spectral lines that are proportional to l^2 rather than the usual linear dependence. This may be called Yangian abnormal Zeeman effect in H-A.

To conclude this section we would like to state that in preserving all the known results for H-A there are democracy between **L** and **J** given by Eq. (5.1). The later gives rise to new effect regarding the abnormal intensities of spectrum lines at large n and l even though very small.

ACKNOWLEDGMENTS

Mo-Lin Ge thanks M. Batchelor, P. A. Pearce and V. Bazhanov for their great hospitality extended to him during the workshop in Honor of Prof. R. Baxter. This work is in part supported by NSF of China.

REFERENCES

1. V. Drinfel'd, *Sov. Math. Dokl.* **32**:254 (1985).
2. V. Drinfel'd, *Quantum Group* (PICM, Berkeley, 1986), p. 269.
3. V. Drinfel'd, *Sov. Math. Dokl.* **36**:212 (1985).
4. A. Belavin, *Phys. Lett. B* **283**:67 (1992).
5. M. Jimbo (ed.), *Yang–Baxter Equations in Integrable Systems* (World Scientific, Singapore, 1990).
6. R. Baxter, *Exactly Solvable in Statistical Mechanics* (Academic, London, 1982).
7. P. P. Kulish, I. K. Sklyanin, *Lecture Notes in Physics* **151**:1 (1982).
8. V. E. Korepin, N. M. Bogoliubov, and A. G. Izergin, *Quantum Inverse Scattering Method and Correlation Function* (Cambridge Univ. Press, Cambridge, 1993).
9. V. Chari, A. Pressley, Yangian and *R*-matrix, *L'Enseignement Mathématique* **36**:267–302 (1990); *A Guide to Quantum Groups* (Cambridge Univ. Press, Cambridge, 1994).
10. F. D. M. Haldane, *Phys. Rev. Lett.* **60**:635 (1998).
11. S. Shastry, *Phys. Rev. Lett.* **60**:639 (1988).
12. F. D. M. Haldane, Physics of the ideal semion gas: spinions and quantum symmetries of the integrable Haldnade–Shastry spin–chain, in *Proc. 16th Taniguch Symp. on Condensed Matter* (Springer, Berlin, 1994).
13. D. B. Uglov and V. E. Korepin, *Phys. Lett. A* **190**:238 (1994).
14. F. Gömann and V. Inozemtsev, *Phys. Lett. A* **214**:161 (1996).
15. M. L. Ge, K. Xue, and Y. M. Cho, *Phys. Lett. A* **260**:484 (1999).
16. M. L. Ge, K. Xue, and Y. M. Cho, *Phys. Lett. A* **249**:358 (1998).
17. V. I. Anol'd and S. D. Novikov, *Encyclopaedia of Mathematical Sciences*, *Vol. 4*, *Part IV: Dynamical Systems* (Springer, Berlin, 1985) pp. 180–183.
18. T. T. Wu and C. N. Yang, *Nucl. Phys. B* **107**:365 (1976).